Variance vs. Standard Deviation: Variability Reduction Through Operations Reversal

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We show that if the analysis of the model of Lee and Tang used standard deviation rather than variance, some nonintuitive predictions of their analysis would be eliminated.
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1. Introduction

In their paper "Variability Reduction Through Operations Reversal in Supply Chain Re-Engineering," Lee and Tang (1998) model a two-stage supply chain, where two operations (A and B) may be performed in either order (A-B or B-A). Operation A introduces a features and operation B, independently b features resulting in ab variations of final product.

The two-stage system is operating as a pull-system. Based on the fact that "variability drives buffer inventory," the authors examine the variability at each of the stages of the process. Since in the considered system each of the demand streams is transferred (immediately or after information lead time) to higher echelon(s), the requirements for both the raw materials and final products will be the same and the only place that experiences a different stream of demand is the first stage (producing intermediate products).

Following the paper's notation, let X_{ij} denote the final demand for choice i of operation A and feature j of operation B. The authors compare the two alternatives (A-B and B-A) by comparing sums of variances of demands faced by the first stage:

$$\mathrm{Var}(X_{11} + \cdots + X_{1b}) + \cdots + \mathrm{Var}(X_{a1} + \cdots + X_{ab})$$

with

$$\mathrm{Var}(X_{11} + \cdots + X_{a1}) + \cdots + \mathrm{Var}(X_{1b} + \cdots + X_{ab}).$$

If the first sum is smaller then operation A should be performed first.

The total demand is assumed to have a general distribution with mean μ and standard deviation σ. For each unit of demand, the choice of features is random: Features i and j are requested with probability θ_{ij} (where $\sum_{ij} \theta_{ij} = 1$), and choices are independent from unit to unit. In the simplest scenario with two features for each operation ($a = b = 2$), where portion p is for feature 1 in stage A, and portion q is for feature 1 in stage B, we have $\theta_{11} = pq$, $\theta_{12} = p(1 - q)$, $\theta_{21} = (1 - p)q$, and $\theta_{22} = (1 - p)(1 - q)$.
The authors show that A-B has smaller variance than B-A if

$$(\mu - \sigma^2)[p(1 - p) - q(1 - q)] < 0.$$

In another version of the model where the numbers of features in stages A and B are a and b and each feature has the same probability, the authors show that A-B is preferable if

$$(\mu - \sigma^2)\left(\frac{-1}{a} - \frac{-1}{b}\right) < 0.$$

All of the conditions in the paper that determine which operations should be performed first include (as above) the term $(\mu - \sigma^2)$. If $\mu - \sigma^2 > 0$, both of the equations above confirm our intuition: Choose the less variable operation ($|.5 - p| \geq |.5 - q|$), or the operation with fewer potential features ($a \leq b$) as the

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first one. Similarly, other conditions included in the paper confirm our intuition when $\mu - \sigma^2 > 0$.

The presence of $(\mu - \sigma^2)$ is surprising and potentially misleading. It may suggest that for *any* distribution the benefits of reversal of operations depend on whether $\mu - \sigma^2 > 0$ (<0).

In this note we examine why this counter-intuitive result takes place and compare it with a result from an alternative formulation.

2. Analysis

We first briefly examine two issues influencing the insights from the paper of Lee and Tang (1998): (i) size and variability of demand, and (ii) equivalent objectives.

(i) Notice first that for $\mu - \sigma^2 < 0$, results are nonintuitive: (with everything else equal) we would rather have an operation introducing 100 features ($a = 100$) first rather than a nondifferentiating operation ($b = 1$). Note that even a relatively low coefficient of variation may cause $\mu - \sigma^2 < 0$.

(ii) It is possible to reinterpret the objective function used in the model in two equivalent ways. For clarity of presentation, this simple result is shown for $a = b = 2$, but it holds for general a and b.

FACT 1. *(a) If* $\mathrm{Cov}(X_{11}, X_{12}) + \mathrm{Cov}(X_{21}, X_{22}) < \mathrm{Cov}(X_{11}, X_{21}) + \mathrm{Cov}(X_{12}, X_{22})$, *then A should be performed first. (b) If* $\mathrm{Cov}(X_{11} + X_{12}, X_{21} + X_{22}) > \mathrm{Cov}(X_{11} + X_{21}, X_{12} + X_{22})$, *then A should be performed first.*

Based on (a), we can think about the choice of operations in terms of choosing a partition of all demands among feasible partitions that allows us to combine negatively correlated demands. Based on (b), however, the choice is at the cost of larger covariance between $(X_{11} + X_{12})$ and $(X_{21} + X_{22})$. Thus, the potential decrease of the sum of the variances is at the cost of having two streams of highly correlated demands. While Lee and Tang do mention the importance of correlation, they do not provide any rigorous way of dealing with it.

The results described above could be caused either by probability structure or by objective function. The next section shows that, with the same probability structure and modified objective, the surprising behavior disappears.

3. Using Standard Deviation

Costs in a manufacturing system may be driven by many factors, one of them is variability. One of the important costs, cost of inventory, is strongly tied to variability faced by the system. In the case of normal distributions, with linear holding and stock-out costs, an up-to-level policy is optimal and the ideal inventory level is usually expressed as $\mu + z * \sigma$. Given z, the cost is linear in σ. A recent paper by Gallego (1998) also shows that the optimal costs for (Q, R) policy can be bounded (from above) by a function that is proportional to standard deviation. Models with fixed ordering costs may potentially be the place where cost is a nonlinear (a higher power) function of standard deviation. Ehrhardt (1979), for example, in his computational study of optimal (s, S) policies, shows that $S - s = C_0 + C_1 * \sigma^{0.1382}$ and $s = D_0 + D_1 * \sigma^{1.206}$ approximate the optimal values very well.[1] This suggests that the inventory cost in this case is also proportional to $\sigma^{1.2}$. The references above also suggest that the inventory costs may be described more appropriately using standard deviation rather than variance.

In general, minimizing the sum of variances is not trivially equivalent to minimizing the sum of standard deviations. It is at present unclear to us what kinds of costs are better modeled by using the sum of variances rather than the sum of standard deviations. Given that, at least in some of the situations, the sum of standard deviations may be more appropriate, we examine the behavior of the model in such a case.

FACT 2. *Under all the assumptions of the model described in Lee and Tang (1998).*

$$Std(X_{11} + X_{12}) + Std(X_{21} + X_{22}) \text{ is concave in } p,$$

where $Std(X)$ *is the standard deviation of* X *and* p *is the probability of feature* 1 *in stage A.*

[1] C_0, C_1, D_0, and D_1 are functions of mean demand, lead time, fixed cost, holding cost, and penalty cost.

PROOF.

$$\tilde{C}(p) = \text{Std}(X_{11} + X_{12}) + \text{Std}(X_{21} + X_{22})$$
$$= \sqrt{\mu p(1-p) + p^2\sigma^2}$$
$$+ \sqrt{\mu p(1-p) + (1-p)^2\sigma^2}$$
$$= \sqrt{h_1(p)} + \sqrt{h_2(p)}.$$

Since $(\sqrt{h})'' = [2h''h - (h')^2]/4(\sqrt{h})^3$ and $2h_i''(p)h_i(p) - (h_i'(p))^2 = -\mu^2$ for $i = 1, 2$, we have

$$\tilde{C}''(p) = -\mu^2\left[\frac{1}{4(\sqrt{\mu p(1-p) + p^2\sigma^2})^3} + \frac{1}{4(\sqrt{\mu p(1-p) + (1-p)^2\sigma^2})^3}\right]$$
$$< 0. \quad \square$$

FACT 3. *Irrespective of relative values of μ and σ, operation A should be performed first if $|.5 - p| > |.5 - q|$.*

FACT 4. *Assume multiple identical choices, i.e., $p_1 = \cdots = p_a$ and $q_1 = \cdots = q_b$. If $a < b$, then operation A should be performed first.*

PROOF. $\sum_{i=1}^{a} \sqrt{\mu p_i(1-p_i) + \sigma^2 p_i^2}$

$$= a\sqrt{\mu(1/a)(1-1/a) + \sigma^2(1/a^2)}$$
$$= \sqrt{\mu(a-1) + \sigma^2}$$

and clearly smaller a is better. $\square$

Facts 3 and 4 show that the dependence on the term $(\mu - \sigma^2)$ disappears when we minimize the sum of standard deviations rather than the sum of variances, thus eliminating the potentially counterintuitive insights of Lee and Tang.

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