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Statistical Process Control using Shewhart Control Charts with Supplementary Runs Rules

M. V. Koutras · S. Bersimis · P. E. Maravelakis

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Abstract The aim of this paper is to present the basic principles and recent advances in the area of statistical process control charting with the aid of runs rules. More specifically, we review the well known Shewhart type control charts supplemented with additional rules based on the theory of runs and scans. The motivation for this article stems from the fact that during the last decades, the performance improvement of the Shewhart charts by exploiting runs rules has attracted continuous research interest. Furthermore, we briefly discuss the Markov chain approach which is the most popular technique for studying the run length distribution of run based control charts.

Keywords Statistical process control · Control charts · Shewhart · Runs rules · Scans rules · Patterns · Markov chain · Average run length

AMS 2000 Subject Classification 62N10

1 Introduction

In the area of statistical process control (SPC) the most popular technique for maintaining process control is control charting. Control charts are systematically used by practitioners to monitor one or more variables (quality characteristics) that are directly or indirectly related to the production process. Generally speaking, to maintain a control chart we choose samples of outgoing products, calculate an appropriate statistic and plot it in a chart equipped with border lines that are used to determine whether the process is statistically "in-control."

Statistical process control is used to secure, that the quality of the final product will conform to predefined standards. In any production process, regardless of how carefully it is maintained, a certain amount of natural variability will always exist. A process is said to

M. V. Koutras (✉) · S. Bersimis · P. E. Maravelakis
Department of Statistics, University of Piraeus, Insurance Science,
80 Karaoli & Dimitriou Str., Piraeus 18534, Greece
e-mail: mkoutras@unipi.gr

be statistically "in-control" when it operates with only chance causes of variation. On the other hand, when assignable causes are present, then we say that the process is statistically "out-of-control."

In general, control charts are easily implemented in any type of process; this is the main reason they are extensively used nowadays to supervise the quality of the process or of the final product. The most popular uses of control charts are for monitoring the mean or the variance of an adequately selected variable affecting the quality of a process or a product.

The control charts can be classified into several categories, according to several distinct criteria. Depending on the number of quality characteristics under investigation, we may define univariate control charts (for monitoring one process characteristic) or multivariate control charts (for more than one characteristics). In the latter case we should distinguish between control charts for correlated measurements and control charts for independent ones. Another criterion stems from the process evolution; more specifically, if the successive samples are dependent we have control charts for autocorrelated processes while if the successive samples are independent we may define control charts for non-autocorrelated processes. Furthermore, the quality characteristic of interest may be a continuous random variable or alternatively a discrete attribute.

As far as the statistical basis of a control charting procedure is concerned we may partition control charts to the following three main categories: Shewhart, Cumulative Sum (CUSUM) and Exponentially Weighted Moving Average (EWMA) control charts. Among them the oldest and probably the most popular ones are the Shewhart control charts, named after Shewhart (1931) who was the first to introduce them. These charts have a very clear statistical basis, since they are closely associated to the classical statistical hypothesis testing. The CUSUM charts were developed by Page (1954) using Wald's sequential testing theory. Finally, the EWMA charts, which assign larger weight to the most recent observations, were defined by Roberts (1959).

Each of the aforementioned categories of control charts has specific advantages and disadvantages. A Shewhart chart uses the information contained in the most recently inspected sample; as a consequence, it is not very efficient in detecting gradual or small shifts in a process characteristic. In contrast, this type of control chart may instantly detect a large shift in the process level and for this reason it has been used for well over the last 70 years. On the contrary, CUSUM and EWMA control charts are more sensitive in detecting small shifts in a process since they use information from a long sequence of samples.

The insensitivity of the Shewhart control chart in detecting gradual or small shifts was realized at the very beginning; a remedial action suggested to overcome this handicap was the use of supplementary rules based on runs and scans.

During the last decades, the performance improvement of the Shewhart control charts has attracted continuous research interest, although the CUSUM and EWMA control charts seem to have better performance. This may be possibly attributed to the fact that the calculations needed for CUSUM and EMMA charts are rather intricate for shop floor workers.

In this paper we review the recent advances in the area of control charts with supplementary runs rules. In Section 2, we present the basic information about the different phases of control charting as well as the main characteristics of Shewhart control charts. The use of runs rules in Shewhart control charts is presented in Section 3. In Section 4, we discuss a method to calculate the run length distribution of a Shewhart type control chart supplemented with runs rules, by the aid of a Markov chain approach. Finally, some concluding remarks are given in Section 5.

2 The Shewhart Control Charts

2.1 Phases of Control Charting Practice

In control charting practice, two distinct phases have been used in the literature: Phase I and Phase II (see, e.g. Woodall (2000)). In Phase I, the basic aim is to test historical data for identifying whether they were sampled from an in-control process or not. The main tool used in this phase to facilitate the practitioner achieve the previous stated objective is control charting. When the process has been brought to an in-control state, we proceed to the assessment of the in-control parameters that will be subsequently used to decide if the process remains in-control. However, we have to stress that, in this phase, an in-depth study of the process behavior should be carried over before shifting to the next step.

In Phase II, our basic aim is to test future data for specifying whether our process remains in-control or has shifted to the out-of-control state. The most commonly used method for making this decision is by launching a control chart scheme. Every new observation or subgroup is tested and by the aid of specific rules we decide whether the state of our process has changed or not.

As Woodall (2000) pointed out, much work, including process understanding and process improvement, is often required in the transition from Phase I to Phase II.

2.2 Basic Characteristics of a Univariate Shewhart Control Chart

Shewhart control charts have been proposed for both Phases I and II. In what follows, we present the statistical concepts that form the basis of Shewhart control charts for continuous variables (control charts for discrete variables are established in a similar fashion).

Consider an industrial process that is running continuously. A standard quality control plan requires sampling one or more items from this process periodically and making the appropriate quality measurements. Usually, more than one item is measured at each time point, say t, to ensure accuracy. Then, the quality control engineer, would like to have at hand an easy to apply technique that will help him decide whether the process mean or variance has shifted away from a reference value. Let us first assume that we are interested in controlling the process mean.

A Shewhart control chart is a graphical display of the product or process quality characteristic (that has been measured or computed from a sample) versus the sample number or time t. The basic characteristics of a univariate Shewhart chart are the Center Line (CL), the Upper Control Limit (UCL), and the Lower Control Limit (LCL). The most popular choices for these quantities are

$$\mathrm{UCL} = E(W) + 3 \times \sqrt{\mathrm{Var}(W)}, \mathrm{CL} = E(W), \mathrm{LCL} = E(W) - 3 \times \sqrt{\mathrm{Var}(W)},$$

where $W = g(\boldsymbol{X})$ is a statistical function of the process data vector $\boldsymbol{X}$ that is used for estimating the process mean, $E(W)$ is the mean value of W and $\mathrm{Var}(W)$ is the variance of W. The aforementioned control limits UCL, LCL are usually referred as three-sigma limits. Here, the assumption of independence of the distribution of $W=g(\boldsymbol{X})$ in time has been made. In Phase I, the parameters $E(W)$ and $\mathrm{Var}(W)$ are estimated by appropriate estimators, while in Phase II, the process is checked for detecting if these parameters have changed.

As long as the points resulting by calculating the statistic $W=g(\boldsymbol{X})$ for the new data vector $\boldsymbol{X}$ remain within the control limits (and no patterns signaling presence of an assignable cause occurs) the process is declared to be in-control, and no action is deemed necessary. A plot of a process that is statistically in-control is given in Fig. 1.

In the foregone analysis we used upper and lower control limits that are placed three standards deviations away from the mean of the test statistic W. The rationale for this is that, when W follows a normal distribution, the resulting interval (LCL,UCL) has probability 0.9973 of including the values of W, as long as the process is in-control. Therefore, the probability of declaring the process out-of-control while it is actually in-control is very low (only 0.0027). The above situation is met for example when W is the average of normal measurements. However, even if the individual measurements are non-normal, it follows by the central limit theorem that the sample averages should have a distribution that is roughly normal and therefore it would be unlikely as well to differ from its mean by more than three standard deviations.

It seems plausible to assume that the presence of an assignable cause of variation, will affect the behavior of the statistic $W=g(\boldsymbol{X})$ plotted on the control chart. Control charts should be capable to create an alarm when a shift in the level of one or more parameters of the underlying distribution occurs or a non-random behavior comes into. Normally, such a situation will be reflected in the control chart by points plotted outside the control limits (see Fig. 2) or by the presence of specific patterns.

The most common non-random patterns are (a) cycles, (b) trends, (c) mixtures and (d) stratification. Figure 3 presents a control chart of a process with a cyclical behavior while Fig. 4 depicts a control chart where a trend component affects the mean level of the process. Figure 5 displays the control chart for a process with a stratification problem, and finally, Fig. 6 the control chart of a process with a mixture problem.

2.3 Performance Evaluation of Shewhart Control Charts

The performance of a control chart is usually evaluated by the Average Run Length (ARL). The run length may be viewed as the waiting time until the first occurrence of an event ε creating an out-of-control alarm, that is, the number of samples up to and including the sample where the first formulation of ε was detected.

The in-control ARL (ARL_{in}) of a Shewhart chart is the average number of samples plotted on a control chart until a possible out-of-control state is signaled, while the process

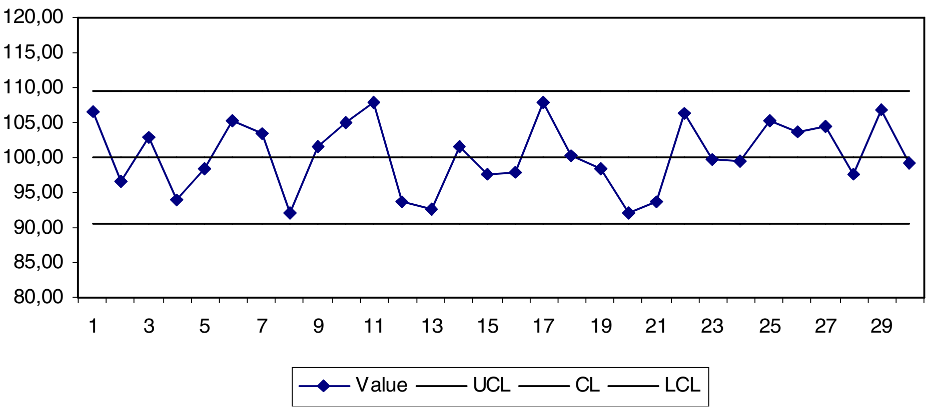

Fig. 1 Shewhart control chart for controlling the mean level of a process

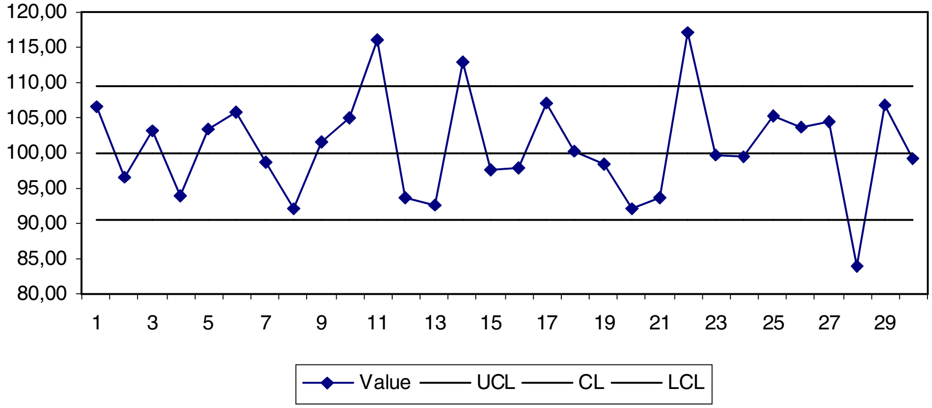

Fig. 2 Control chart of a process subject to several shifts

is actually in-control. The out-of-control ARL (ARL_{out}) is the average number of samples plotted in a control chart before an out-of-control state is signaled while the process under study is actually out-of-control.

2.4 Out-of-control Conditions and Analysis of Patterns on Control Charts

All the processes in Figs. 2, 3, 4, 5, and 6 are out-of-control. In Fig. 2, the control chart identified a possible out-of-control shift (points 22 and 28 are outside the control limits) while in Figs. 3, 4, 5 and 6 the corresponding control charts were not able to detect an abnormal condition (all the points are inside the control limits). In Fig. 3 there is a consistent change in the mean level of the process parameter under consideration which was not recognized by the control chart in an early stage. A similar situation occurs in Figs. 4, 5 and 6 in the sense that the plotted points on the control charts exhibit a nonrandom behavior and therefore they should be treated as points from an out-of-control

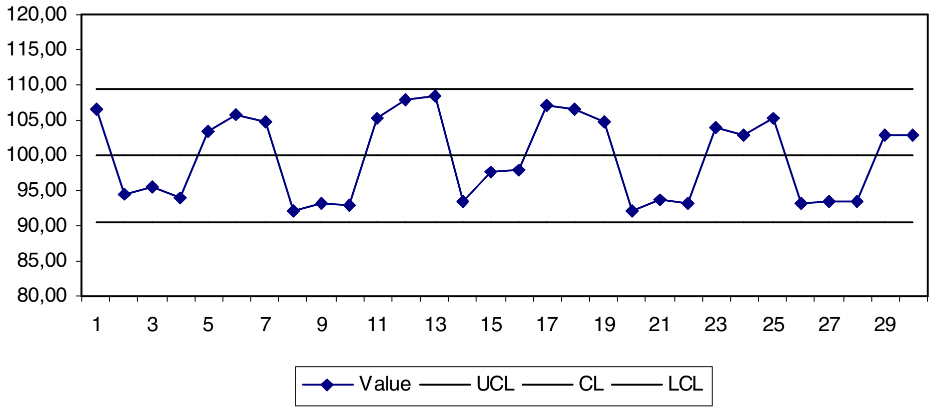

Fig. 3 Control chart of a process with a cyclical component

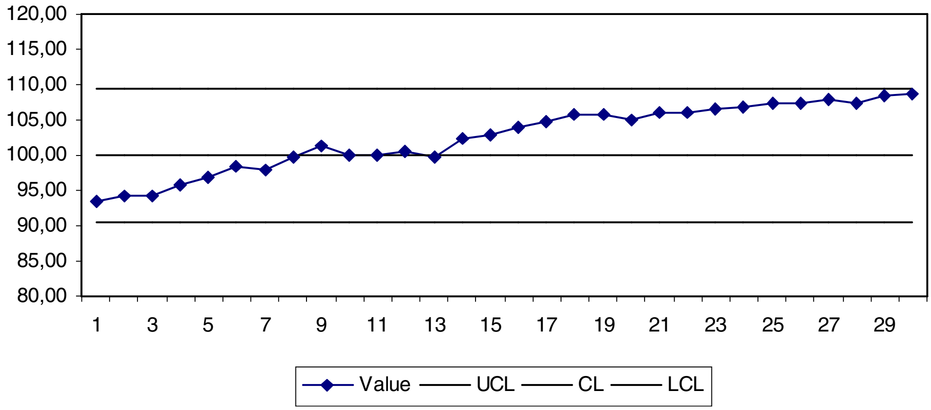

Fig. 4 Control chart of a process with a trend component

process. The control chart of Fig. 2 succeeded in identifying the out-of-control condition because a large mean's shift occurred, while the gradual or cyclical shift of the process mean (Figs. 3, 4, 5 and 6) did not trigger an alarm for abnormal behavior through the established control chart. To overcome this problem several sensitizing criteria based on runs and scans rules have been suggested for use with standard Shewhart control charts. This will be discussed in the next section.

3 Shewhart Control Charts with Supplementary Runs Rules

3.1 The Concept of Runs in SPC and the Western Electric Company Runs Rules

A run may be defined as an uninterrupted sequence of alike elements bordered at each end by elements of different type (for a detailed presentation of the theory of runs and related literature, the interested reader may consult the recent monograph by Balakrishnan and Koutras (2002)).

The use of runs in quality control goes back to the 1940s. At this period, several authors suggested quality control plans that use as acceptance / rejection criterion the occurrence of prolonged sequences of points that fall within/outside the control limits, while others supplemented the basic criteria with additional runs rules (see, e.g., Mosteller (1941); Wolfowitz (1943)).

Later on, the Western Electric (1956) proceeded to a systematic presentation of a set of decision rules based on runs and scans, reinforcing the worldwide use of runs in the field of SPC. The main objective for suggesting this set of rules was to increase the sensitivity of the Shewhart charts and improve their potential to detect non-random patterns.

Before presenting these rules it is necessary to introduce two new sets of control limits which will be termed warning limits. The first set consists of the Upper Warning Limit (UWL) and the Lower Warning Limit (LWL) which are given by

$$\mathrm{UWL} = E(W) + 2 \times \sqrt{\mathrm{Var}(W)}, \quad \mathrm{LWL} = E(W) - 2 \times \sqrt{\mathrm{Var}(W)}$$

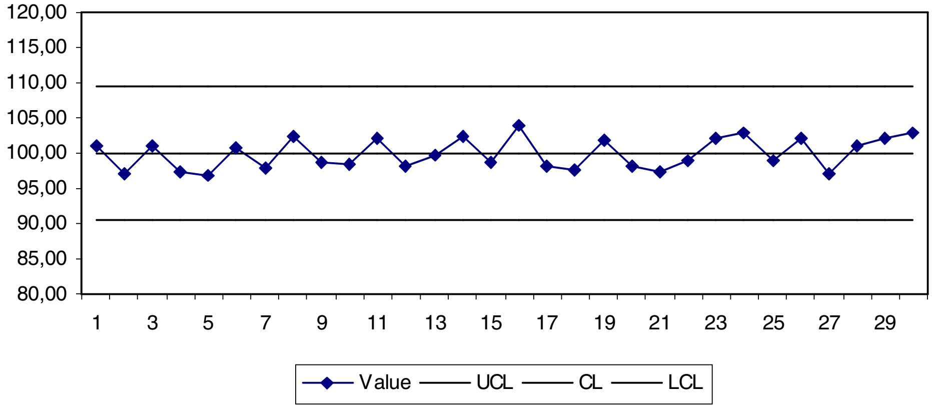

Fig. 5 Control chart of a process with a stratification problem

(these limits are usually called two-sigma warning limits). The second includes an Upper Inner Warning Limit (UIWL) and a Lower Inner Warning Limit (LIWL) given by

$$\text{UIWL} = E(W) + \sqrt{\text{Var}(W)}, \quad \text{LIWL} = E(W) - \sqrt{\text{Var}(W)}$$

(these limits are usually called one-sigma warning limits).

The four additional limits give birth to six disjoint zones in the area between the two control limits; These zones have been labeled as A_U, B_U, C_U, C_L, B_L, A_L in Fig. 7 and have been extensively used by many researchers to define appropriate runs rules that decrease the out-of-control ARL without affecting the in-control ARL.

According to the Western Electric (1956), an out-of-control signal should be produced whenever at least one of the following events occurs:

1. One point plots outside the three-sigma control limits;
2. Two out of three consecutive points plot beyond the two-sigma warning limits;
3. Four out of five consecutive points plot at a distance of one-sigma or beyond from the center line;
4. Eight consecutive points plot on one side of the center line.

It should be stressed that, the above rules are applied to one side of the center line at a time; therefore, a point above the UWL followed by a point below the LWL does not create a violation of rule 2 and will not signal an out-of-control signal. It is clear that the stock of such rules is inexhaustible. Page (1955), Roberts (1958), Bissel (1978) also suggested several additional runs rules.

According to Montgomery (2001) the most widely used runs rules by quality control practitioners are the abovementioned Western Electric rules supplemented by the following six additional rules:

5. Six points in a row steadily increasing or decreasing;
6. Fifteen points in a row in zones C_U and C_L (see Fig. 7);
7. Fourteen Points in a row alternating up and down;
8. Eight points in a row on both sides of the center line with none in Zone C (see Fig. 7);
9. An unusual or non random pattern in the data;
10. One or more points near a warning or control limit.

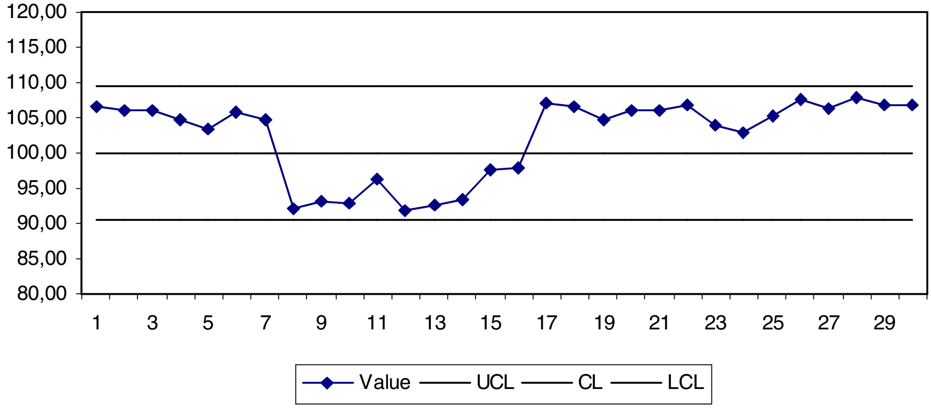

Fig. 6 Control chart of a process with a mixture problem

It should be stressed that not all the runs rules mentioned above involve pure runs in the sense given in the beginning of this paragraph. Some of them are triggered when a proportion of points within a specific fixed length window falls outside the predetermined limits (such a pattern is usually called scan or generalized run - see e.g. the monographs by Balakrishnan and Koutras (2002) or Glaz et al. (2001)) while others look at runs up and down or even at more complicated unusual patterns.

3.2 Review of Shewhart Control Charts with Supplementary Sensitizing Runs Rules

The case of control charts with warning limits was first extensively studied by Page (1955) who proposed four runs rules. He also proceeded to a systematic study of the chart's performance by introducing a Markov chain approach for calculating the exact run length distribution. Earlier, Mosteller (1941), Dudding and Jannet (1942) and Weiler (1953) had discussed the case of warning limits and runs rules, with the latter treating the case of control charts with warning limits only. Moore (1958) used rules similar to the ones given by Page (1955).

Roberts (1958) commented that the use of runs increases the number of false alarms and tossed for first time the term zone chart to describe a control procedure that makes use of runs rules in a Shewhart chart (zone charts will be described later on). Westgard and Groth (1979) and Westgard et al. (1981) documented the intuitive fact that runs rules increase the false alarm rates. Their work was justified through numerical examples.

Wheeler (1983) computed the power of the one-sided Shewhart chart supplemented by four runs tests for controlling the mean of a process. He concluded that the enhanced power of the runs rules to detect small shifts in the process is unquestionable while Nelson (1984) described in detail the advantages and the disadvantages of Shewhart charts using runs rules. He suggested that much care should be exercised when using several decision rules simultaneously because multiple decision rules affect drastically the calculations of the probabilities α,β (type I and II errors, respectively); as a consequence, the laws governing the final decision, may become so intricate as to preclude a complete analysis of the ARL calculations. To make this more transparent assume that someone uses h decision rules with rule i having probability α_{i} that W plots beyond the control limits (under the assumption that the process is in-control). Then, the overall probability of false alarm for the decision based on all h criteria equals $\alpha = 1 - \Pi_{i=1}^{h}(1 - \alpha_i)$, provided that all h decision rules are independent. Unfortunately this is not valid in the case of the usual sensitizing rules

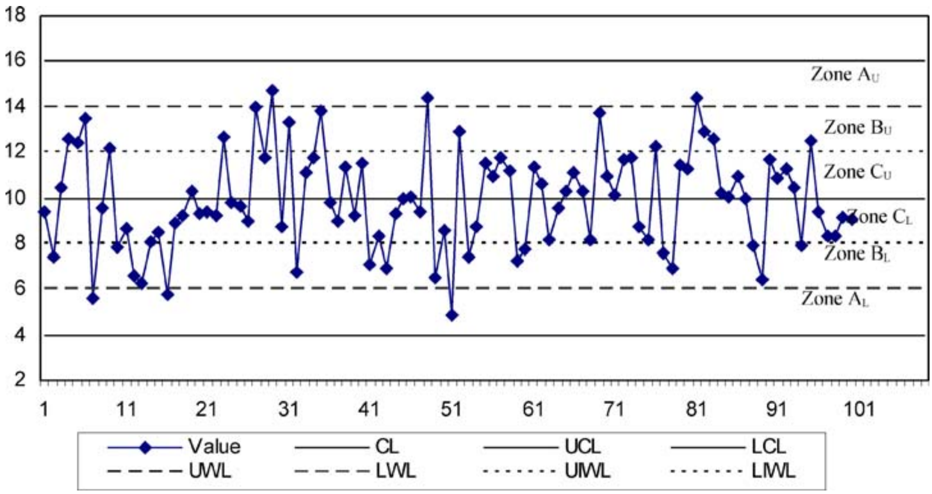

Fig. 7 Shewhart chart for controlling the mean level of a process supplemented with warning limits

(Nelson (1985); Montgomery (2001)), and therefore the evaluation of α becomes quite involved.

In order to describe the control charting techniques that use additional rules in a uniform way, Champ and Woodall (1987) introduced a general class of run rules, as follows: an out-of-control signal is given if k of the last m standardized sample statistics W' fall within the interval (a,b) where $k \le m$ and $a \le b$ (notation: $W'(k,m,a,b)$). According to their notation, the Western Electric Company's runs rules can be expressed as follows:

Rule 1: $C_1 = \{W'(1,1,-\infty,-3), W'(1,1,3,\infty)\}$	Rule 2: $C_2 = \{W'(2,3,-3,-2), W'(2,3,2,3)\}$
Rule 3: $C_3 = \{W'(4,5,-3,-1), W'(4,5,1,3)\}$	Rule 4: $C_4 = \{W'(8,8,-3,0), W'(8,8,0,3)\}$

The same authors presented three additional runs rules, which are frequently used in British industry practice. Using their notation, one may denote by $C_{ij\ldots k} = C_i \cup C_j \cup \ldots \cup C_k$, the composite Shewhart control chart, using $C_i, C_j, \ldots C_k$ as criteria for detecting an out-of-control process. As Champ and Woodall (1987) concluded, when the power of the control chart is increasing, the false alarm rate is also increasing.

Palm (1990) provided tables, with the ARL values and the percentiles of the run length distribution of Shewhart control charts supplemented with the most popular rules. Champ and Woodall (1990) published a computer program that uses a Markov chain approach to obtain exact probabilities in the case of pre-specified limits, while Walker et al. (1991) conducted a simulation study in order to asses the false alarm rate of Shewhart charts with supplementary runs rules. The use of simulation was indispensable, since they studied the case where control limits are not pre-specified but are estimated from available data. In this case, which is of special interest for quality practitioners, the exact calculations although feasible, quickly became tedious. They compared their results with the results of Champ and Woodall (1987) drawing analogous conclusions.

Champ (1992) provided a study for the steady-state run length distribution of a Shewhart chart supplemented with runs rules. In contrast to the most studied "initial-state run length distribution," which is defined as the number of the sampling stages at which the chart creates an out-of-control alarm for the first time given it begins in some initial state, the "steady-state run length distribution" is defined to be the number of sampling stages until a signal is given after the procedure has reached a steady-state.

Alwan et al. (1994) studied the ARL of Shewhart type control charts with supplementary runs rules in the presence of autocorrelation.

Lowry et al. (1995) showed that the performance of a Shewhart chart for controlling the dispersion of a process, when supplemented with the standard Western Electric Company runs rules, is poor. For this reason they suggested alternative rules in order to obtain a performance comparable to the classical Shewhart chart for controlling the mean of a process, with the standard Western Electric Company runs rules applied on it.

Das and Jain (1997) as well as Das et al. (1997) developed economic design models for Shewhart type control charts supplemented by runs rules.

Champ and Woodall (1997) considered a compound rule (set of runs rules) supplementing a Shewhart control chart and calculated the signal probabilities of each one of the single runs rules comprising the compound rule. These signal probabilities are useful in order to choose runs rules using as decision criterion the false alarm rates.

Shmueli and Cohen (2003) introduced a method for calculating the run length distribution of run based control charts by exploiting probability generating function arguments. Using their method, they compared the entire run length distribution for various popular runs rules. As they commented, the advantage of taking into account the entire distribution, instead of using only the expectation, is that more detailed conclusions may be drawn. Fu et al. (2002, 2003) introduced a unified Markov chain approach for computing the run length distribution in control charts with simple or compound rules. Khoo (2003) provided a detailed description of methods for designing control charting schemes with supplementary runs rules as well as a presentation of Markov chain techniques for evaluating the run length distribution.

Hurwitz and Mathur (1992), Derman and Ross (1997), Klein (2000), and Antzoulakos and Rakintzis (2006) applied more sophisticated pattern based rules, for improving the performance of Shewhart charts.

Wludyka and Jacobs (2002) applied runs rules in charting procedures of a multi-stream binomial process. They used the Markov chain approach to calculate the performance of the proposed procedure. The case of multi-stream processes was also studied by Nelson and Stephenson (1996), while Kuralmani et al. (2002) discussed the use of runs in high yield processes.

3.3 The Performance of Runs Based Shewhart Charts in Detecting Linear Trend

During the last two decades, a number of authors have dealt with the problem of establishing dedicated rules for detecting linear trend in a Shewhart chart. Aerne et al. (1991) compared the performance of the most popular control charts of this type in the presence of linear trend in the process. They concluded that CUSUM and EWMA control charts have better performance than the Shewhart charts with or without runs rules.

Davis and Woodall (1990) studied the r in a row up or down rule, and concluded that it is ineffective in detecting a trend in the process mean. Divoky and Taylor (1995) used simulation to study the performance of a large number of runs rules. More specifically, they retrieved the full range of scan rules of the type r out of k and provided heuristic suggestions for the selection of the best one.

3.4 Modifications of Univariate Shewhart Charts Using Runs Rules

Jaehn (1987, 1989) proposed a chart, which has the same performance as the Shewhart chart with supplementary runs rules, but is simpler for a practitioner to apply. This special type of chart, which was termed as zone chart (as already mentioned this nomenclature was used for the first time by Roberts (1958)), has eight zones, four on each side of the center line. Scores are assigned to each zone, and the procedure signals an out-of-control condition when the total score exceeds a threshold value.

Davis et al. (1990) improved the performance of the zone chart by modifying the scoring system. Davis et al. (1994) also incorporated a fast initial response feature to this chart. Davis and Krehbiel (2002) studied the performance of Shewhart chart with runs rules against the zone chart when linear trend is present. Their study revealed that, in this case, the zone chart is superior and therefore is preferable for detecting linear trends.

Klein (1997) proposed a modified composite Shewhart-EWMA chart which uses a Shewhart component, requiring two (or more) plotted points beyond reduced width control limits, in order to give an out-of-control signal. Using an extensive simulation study, he suggested schemes with better ARL profiles and fast initial response characteristics than those associated with run based Shewhart schemes.

An important factor of Shewhart control charts is the sampling frequency. A typical Shewhart chart uses fixed sampling intervals (FSI). In the literature, various suggestions have been given, for applying variable sampling intervals (VSI) to a control chart. The Shewhart charts with VSI, perform much better than the typical Shewhart charts with FSI. Amin and Letsinger (1991) used runs rules as a switching mechanism between control procedures of different sample sizes.

Quesenberry (1991) applied runs rules in Q charts. Bourke (1991) developed a modified Shewhart type chart in order to detect a shift in fraction of nonconforming products. This chart is based on the length of runs of conforming products contained between two non-conforming products.

Wu and Spedding (2000a, 2000b), Wu et al. (2001) and Wu and Yeo (2001) proposed the so-called synthetic control chart which consists also a modification of runs rules applied in a Shewhart control chart. Such a chart, as proved by Davis and Woodall (2002), can be represented as a control chart with runs rules having a head start feature. Synthetic control charting, as a statistical process control technique, can be used to detect shifts in the process mean. However, unlike the standard Shewhart charts used in statistical process control, a signal is not generated upon the occurrence of a single point outside of the limits. Instead, when a sample produces a value outside the control limits, the practitioner looks at the number of samples that have been examined since the last time a point fell outside the limits, or since the first sample if no points have been plotted outside the limits before. If this number is sufficiently small, then a signal is generated.

3.5 Multivariate Shewhart Control Charts with Supplementary Runs Rules

Aparisi et al. (2004) investigated the performance of Hotelling's chi-square control chart with supplementary runs rules. Besides the classical out-of-control criterion, they proposed the use of three additional rules based on runs of length 7 and 8 and two out of three scans. They concluded that, for moderate shifts, the combined use of all these supplementary rules, improves the out-of-control ARL value of the chi-square control chart by approximately 25%.

A similar procedure was presented by Khoo and Quah (2003). Using a different approach, Koutras et al. (2006) combined the theory of success runs and Hotelling's chi-square control chart, and arrived at a systematic procedure, which improved the (weak) performance of Hotelling's chi-square control chart in the case of relatively small mean vector shifts. The final numeric results were similar to the results reported by Aparisi et al. (2004).

4 The Run Length Calculation

As already mentioned in the previous paragraphs a commonly used performance indicator for a control chart is its ARL. If T represents the number of samples drawn until the chart produces an out-of-control signal (given an initial state of the process), then T is usually referred to as initial-state run length. Then the ARL_{in} is simply defined as the mean $E(T)$ of T, provided the process is in-control and represents the average number of samples until the process is declared out-of-control while in fact it is in-control. In a similar fashion, ARL_{out} represents the mean of T provided that the process is out-of-control.

Let us now assume that the decision whether the process has shifted out-of-control is couched on the test statistic $W=g(\boldsymbol{X})$ and denote by $F(\cdot)$ the cumulative distribution function (cdf) of W. If the process is in-control, the cdf will be denoted by $F_{in}(\cdot)$ and the mean and variance of W by μ and σ^2, respectively, (both of them will be assumed known).

In the case of standard Shewhart control charts, which signal as soon as the first point plots outside the control limits LCL,UCL, the probability for a single point to plot outside the control limits is given by

$$p_{in} = 1 - P(\text{LCL} \leq W \leq \text{UCL}).$$

If the control limits are positioned symmetrically around μ, we may express p_{in} in the form

$$p_{in} = 1 - F_{in}(\mu + L\sigma) + F_{in}(\mu - L\sigma)$$

where

$$L = \frac{\text{UCL} - \mu}{\sigma} = \frac{\mu - \text{LCL}}{\sigma}.$$

Since the distribution of T (run length distribution) is a geometric distribution with probability of success p_{in}, the ARL_{in} value equals $\text{ARL}_{in}=1/p_{in}$.

The best scenario for the in-control ARL is to approach infinity. Since this is practically infeasible, it is quite common among the practitioners to use as an in-control ARL value a number between 300 and 500.

If the process is out-of-control the probability a single point to plot outside the control limits is given by

$$p_{out} = 1 - P(\text{LCL} \leq W \leq \text{UCL}) = 1 - F_{out}(\mu + L\sigma) + F_{out}(\mu - L\sigma)$$

and the respective ARL equals $\text{ARL}_{out}=1/p_{out}$. Ideally, we would like the out-of-control ARL to approach unity. However, it is extremely difficult to bring both ARL_{in} and ARL_{out} to pre-specified desirable values. As a consequence, in practice we usually assign a pre-specified fixed value to the in-control ARL and then proceed to the establishment of rules that minimize the out-of-control ARL.

If the random variable $W=g(\boldsymbol{X})$ follows a normal distribution with known parameters μ and σ^2 then

$$p_{in} = 1 - P\left(\mathrm{LCL} \leq W \leq \mathrm{UCL} \middle| W \sim N\left(\mu, \sigma^2\right)\right)$$
$$= 1 - P\left(\mu - L\sigma \leq W \leq \mu + L\sigma \middle| W \sim N\left(\mu, \sigma^2\right)\right)$$

and therefore

$$p_{in} = 1 - \Phi(L) + \Phi(-L) = 2\Phi(-L), \mathrm{ARL}_{in} = \frac{1}{p_{in}} = \frac{1}{2\Phi(-L)}.$$

For the commonly used value $L=3$, we have that $\mathrm{ARL}_{in} \cong 1/0.0027 \cong 370$. If the process shifts from an in-control state to an out-of-control state with mean $\mu^* = \mu + \delta\sigma$, then

$$P_{out} = 1 - P\left(\mathrm{LCL} \leq W \leq \mathrm{UCL} \middle| W \sim N\left(\mu + \delta\sigma, \sigma^2\right)\right) =$$
$$= 1 - P\left(\mu - L\sigma \leq W \leq \mu + L\sigma \middle| W \sim N\left(\mu + \delta\sigma, \sigma^2\right)\right) =$$
$$= 1 - \Phi(L - \delta) + \Phi(-L - \delta)$$

and

$$\mathrm{ARL}_{out} = 1/p_{out} = 1/[1 - \Phi(L - \delta) + \Phi(-L - \delta)].$$

Note that, it is always feasible to determine L in order to achieve a predefined large ARL_{in}; then ARL_{out} will be a function of δ. In general, the rule followed when setting up control charts is to adjust the out-of-control signal rules so that for fixed ARL_{in}, optimal ARL_{out} curves are obtained.

4.1 Calculation of the ARL for Shewhart Control Charts Supplemented with Runs Rules Using a Markov Chain Approach

As already mentioned in the previous sections, several authors (see e.g., Page (1955); Champ and Woodall (1987); Fu et al. (2002, 2003)) have exploited Markov chain techniques to study the run length distribution of control charts.

We shall now briefly illustrate how the Markov chain technique described in the recent papers by Fu et al. (2002, 2003) (see also Fu and Lou (2003)) can be used for the investigation of the run length distribution of control charts incorporating both warning and control limits (see Fig. 7).

Based on the value of the test statistic $W=g(\boldsymbol{X})$ and the specific rules activated for the control chart under study, one may associate the out-of-control signals to the occurrence of specific patterns in a sequence of multistate trials. To this end it is sufficient to label each of the regions formed by the warning and control limits, and describe the out-of-control situation by a family of strings (patterns) ε_a. It is then apparent that the waiting time T until the first occurrence of a pattern belonging to ε_a, describes the time (sample) where an out-of-control signal will be generated.

Let now $\{Y_t, t=0,1,2,...\}$ be a Markov chain defined over a finite state space $\Omega=\{a_1, a_2,...,a_s\}$ so that a_s is an absorbing state and the event $T \leq n$ is equivalent to the event $Y_n=a_s$. The last condition requires that all patterns leading to an out-of-control signal are accumulated to state a_s while $a_1, a_2, ..., a_{s-1}$ will describe all other possible patterns (provided that the Markov property is preserved, one could merge patterns so that the number of states s is minimized).

Finally, let us denote by $\pi_0' = (P(Y_o = a_1), P(Y_0 = a_2), \ldots, P(Y_0 = a_s))$ the (row) vector of initial probabilities of the Markov chain and by $\Lambda = \left[P\left(Y_t = a_j | Y_{t-1} = a_i\right)\right]_{s \times s}$ its transition probability matrix. It is then clear that

$$P(T > n) = 1 - P(Y_n = a_s) = 1 - \pi_0' \Lambda^n e_s$$

where $e_s' = (0, 0, 0, \ldots, 0, 1)_{1 \times s}$ is the unit vector of R^s, while the probability mass function of the waiting time random variable T can be easily derived, on observing that

$$\begin{aligned} P(T = n) = P(T > n-1) - P(T > n) &= P(Y_{n-1} \neq a_s) - P(Y_n \neq a_s) \\ &= \pi_0' \left(\Lambda^n - \Lambda^{n-1}\right) e_s = \pi_0' \Lambda^{n-1} (\Lambda - \mathrm{I}) e_s. \end{aligned}$$

The last formula offers a neat expression for the evaluation of the run length distribution of our control chart. Note also that the probability generating function of the run length distribution can also be easily calculated by the aid of the formula

$$\sum_{n=0}^{\infty} P(T = n) z^n = z\pi_0' (\mathrm{I} - z\Lambda)^{-1} (\Lambda - \mathrm{I}) e_s$$

which is an immediate by-product of the last expression.

In order to elucidate the aforementioned technique let us proceed to presentation of the necessary steps for the calculation of the run length distribution for the rule

$$\begin{aligned} C_{1,5} = C_1 \cup C_5 = \{&W'(1, 1, -\infty, -3), W'(1, 1, 3, \infty)\} \\ &\cup \{W'(2, 2, -3, -2), W'(2, 2, 2, 3)\}. \end{aligned}$$

Figure 8, displays the labels of the zone charts used for reaching to a decision whether the process is in-control or out-of-control. More specifically, we assign the value 3 to both zones (−∞,−3), (3, ∞), the value 2 to zone (−3,−2) and the value 1 to zone (2,3). Finally, zone(−2,2) receives the value 0. Having at hand the corresponding in-control distribution F_{in} (·) of the associated test statistic W=g($\boldsymbol{X}$) we may calculate the probabilities

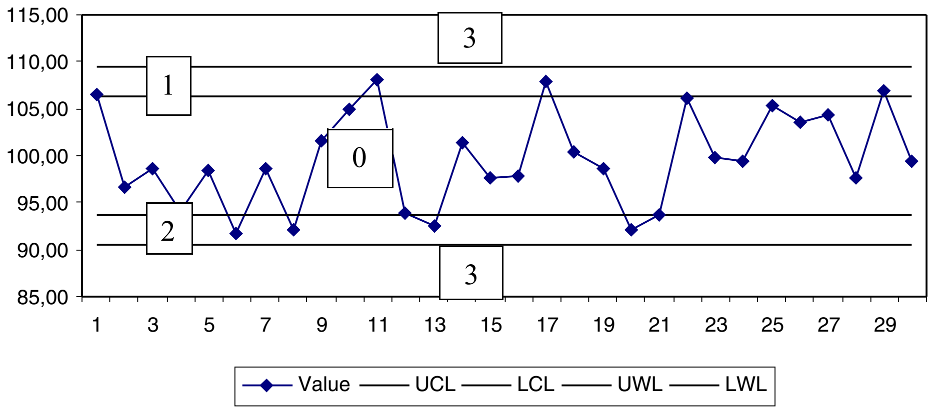

Fig. 8 Zones for a Shewhart control chart with rule $c_{1,5} = c_1 \cup c_5$

$p_0 = F_{in}(2) - F_{in}(-2)$, $p_1 = F(3) - F(2)$, $p_2 = F_{in}(-2) - F_{in}(-3)$, $p_3 = 1 - F_{in}(3) + F_{in}(-3)$. Therefore, provided that the process is in-control, the charting procedure can be modeled by a sequence of multistate trials with four possible outcomes in each trial and respective occurrence probabilities for each outcome p_0, p_1, p_2, p_3.

According to $C_{1,5}$ rule, the events that give an out-of-control signal are: one point plotted in the region (−∞,−3) or one point plotted in the region (3,∞) or two consecutive points in the zone (2,3) or two consecutive points in the region (−3,−2). These events may be described via the multistate model given above by the family of patterns $\varepsilon_a = \{3,11,22\}$. It is now easy to introduce a Markov chain $\{Y_t,\ t=0,1,2,...\}$ on the state space $\Omega = \{0,1,2,\varepsilon_a\}$ where $\varepsilon_a = \{3,11,22\}$ is the absorbing state of the chain, and 0, 1, 2 indicate that the last sample used for monitoring the process, produced a point in zones 0, 1, 2, respectively.

If we denote by $\pi_0' = (P(Y_0 = 0), P(Y_0 = 1), P(Y_0 = 2), P(Y_0 = \varepsilon_a))$ the (row) vector of initial probabilities of the chain and by

$$\Lambda = \begin{bmatrix} & (0) & (1) & (2) & (\varepsilon_a) \\ (0) & p_0 & p_1 & p_2 & 1 - p_0 - p_1 - p_2 \\ (1) & p_0 & 0 & p_2 & 1 - p_0 - p_2 \\ (2) & p_0 & p_1 & 0 & 1 - p_0 - p_1 \\ (\varepsilon_a) & 0 & 0 & 0 & 1 \end{bmatrix}$$

its transition probability matrix, we may readily evaluate the probability distribution of the waiting time T by using the formula $P(\mathrm{T} = n) = \pi_0' \Lambda^{n-1} (\Lambda - \mathrm{I}) e_4$ where $e_4' = (0, 0, 0, 1)$.

The probability generating function of the run length distribution will be given by

$$\sum_{n=0}^{\infty} P(T = n) z^n = z\pi_0' (\mathrm{I} - z\Lambda)^{-1} (\Lambda - \mathrm{I}) e_s$$
$$= \frac{(1 - p_0 - p_1 - p_2)z + (p_1 + p_2 - p_0 p_1 - p_0 p_2 - 2p_1 p_2)z^2 + p_1 p_2 (1 - p_0) z^3}{1 - p_0 z - (p_0 p_1 + p_0 p_2 + p_1 p_2) z^2 - p_0 p_1 p_2 z^3}$$

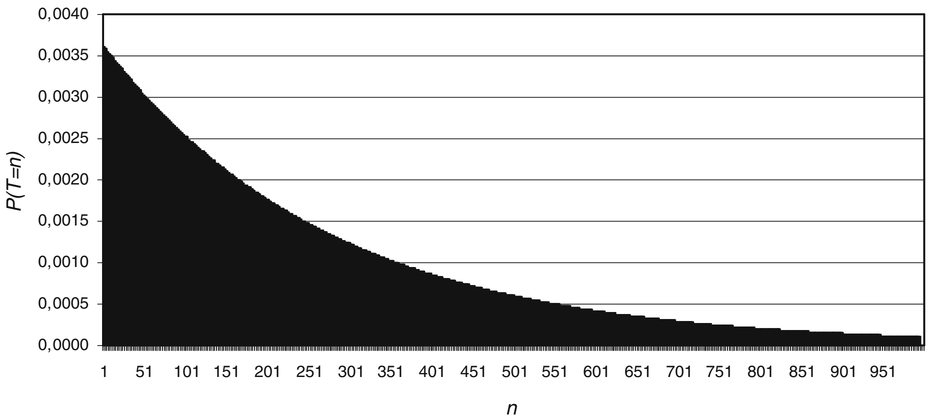

Fig. 9 The in-control run length distribution for the rule $C_{1,5}$

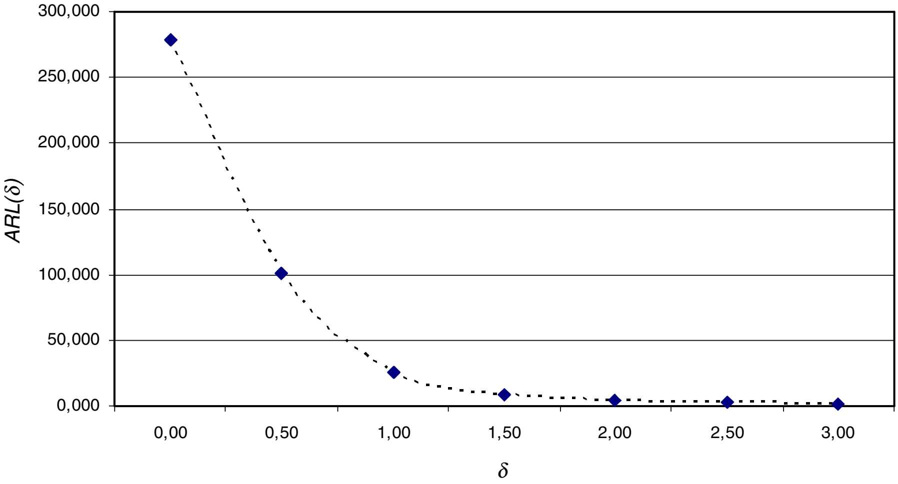

Fig. 10 The *ARL* curve for the rule $C_{1,5}$ as a function of δ

and upon differentiating the last formula with respect to z we may easily obtain the ARL as

$$\mathrm{ARL} = \frac{(1+p_1)(1+p_2)}{1-p_1p_2-p_0(1+p_1)(1+p_2)}$$

Figure 9 displays the run length distribution for the rule $C_{1,5}$ under the assumption that the characteristics we are studying follow the Normal distribution, while Fig. 10 displays the ARL values as a function of the value δ of the shift.

5 Concluding Remarks

The application of runs rules in control charting methodology is a classical technique which has been proved efficient in detecting small and moderate shifts. However, the sensitivity improvement achieved by supplementing the classical control chart by runs rules, has a trade off in the false alarm rate. In simple words, runs rules increase sensitivity but also produce more false alarms.

Nowadays, more efficient charting schemes (e.g., CUSUM, EWMA) have been developed, which offer better performance and therefore, the use of a Shewhart chart with runs rules is not recommended in case of a Phase II analysis. The only unbeatable advantage of Shewhart control charts with runs rules seems to be that they are very easy to use, a fact that makes them attractive for the practitioners even nowadays.

Despite the above criticism, a Shewhart chart supplemented with runs rules is a valuable tool for a Phase I analysis. In this phase, the conditions identifying a statistically in-control process are determined, and a Shewhart chart with runs rules can be beneficially used to spot out and characterize non-random patterns.

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